THE SYSTEM THEORY OF NETWORK CALCULUS

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The Shaper

- **shaper**: forces output to be constrained by $\sigma$
- **greedy** shaper stores data in a buffer only if needed
- examples:
  - constant bit rate link ($\sigma(t)=ct$)
  - ATM shaper; fluid leaky bucket controller
- **Pb**: find input/output relation
A Min-Plus Model of Shaper

**Shaper Equations:**

1. \[ x \leq x \otimes \sigma \]
2. \[ x \leq R \]

- \( R \) and \( x \) are functions
- \( \sigma \) is sub-additive
- \( \otimes \) is min-plus convolution
Network Calculus’s System Theory

- $G = \text{set of functions } Z \rightarrow R^+ \text{ that are wide-sense increasing}$
- Also works in continuous time, functions are left-continuous $R \rightarrow R^+$
- An operator $\Pi$ is a mapping : $G \rightarrow G$
- $\Pi$ is isotone if $x(t) \leq y(t) \Rightarrow \Pi(x)(t) \leq \Pi(y)(t)$
- $\Pi$ is upper-semi continuous iff
  \[ \inf_i(\Pi(x_i)) = \Pi(\inf_i(x_i)) \text{ for } \downarrow \text{ sequences } x_i \]
Min-Plus Linear Operators

- \( \Pi \) is min-plus linear if
  - for any constant \( K \), \( \Pi(x + K) = \Pi(x) + K \)
  - \( \Pi(x \land y) = \Pi(x) \land \Pi(y) \)
  - \( \Pi \) is upper-semi continuous.

- **Representation Theorem**: \( \Pi \) is min-plus linear \( \iff \) there is a unique \( H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+ \), \( \uparrow \) in \( t \) and \( \downarrow \) in \( s \), such that
  \[
  \Pi(x)(t) = \inf_s [H(t,s) + x(s)]
  \]

- min-plus linear \( \Rightarrow \) isotone and upper semi-continuous

- Example: convolution operator
  \[
  C_{\sigma} : x \mapsto \sigma \otimes x
  \]

- Example: \( M \in G \) is given:
  \[
  h_M : x \mapsto y \text{ s.t. } y(t) = \inf_{s \leq t} (M(t) - M(s) + x(s))
  \]
Min-Plus Residuation Theorem

**Theorem:** ([L., Thiran 2001] thm 4.3.1., derived from Baccelli et al., ) Assume that $\Pi$ is *isotone* and *upper-semi-continuous*. The problem

$$x(t) \leq b(t) \land \Pi(x)(t)$$

where $x \in G$ is the unknown function has one maximum solution in $G$, given by

$$x^*(t) = \Pi(b)(t)$$

(Definition of closure)

$$\Pi(x) = \inf \{x, \Pi(x), \Pi\Pi(x), \Pi\Pi\Pi(x), ...\}$$

in other words:

$$x^0 = b; x^i = \Pi(x^{i-1}) \text{ and } x^* = \inf \{x^0, x^1, ..., x^i, ...\}$$
Application to Shaper

There is a maximum solution obtained by iterating

\[ x^{(0)} = R \]
\[ x^{(1)} = R \otimes \sigma \]
\[ x^{(2)} = (R \otimes \sigma) \otimes \sigma = R \otimes \sigma \]

because \( \sigma \otimes \sigma = \sigma \)

Thus \( R^* = \inf (x^{(0)}, x^{(1)}, x^{(2)}, \ldots) = R \otimes \sigma \)

The greedy shaper output is \( R^* = R \otimes \sigma \)

\( C_\sigma \circ C_\sigma = C_\sigma \), the subadditive closure of \( C_\sigma \) is \( C_\sigma \)
Variable Capacity Node

node has a time varying capacity $\mu(t)$
Define $M(t) = \int_0^t \mu(s) \, ds$.
the output satisfies
$R^*(t) \leq R(t)$
$R^*(t) - R^*(s) \leq M(t) - M(s)$ for all $s \leq t$
and is “as large as possible”
Variable Capacity Node

\[ R^*(t) \leq R(t) \]
\[ R^*(t) - R^*(s) \leq M(t) - M(s) \]
for all \( s \leq t \)

- Operator \( h_M \): \( x \mapsto y \) s.t.
  \[ y(t) = \inf_{s \leq t} M(t) - M(s) + x(s) \]

- We have the problem \( R^* \leq R, R^* \leq h_M(R) \)
- \( h_M \circ h_M = h_M \) and the sub-additive closure of \( h_M \) is \( h_M \)
- There is a maximum solution,
  \[ R^*(t) = \inf_{s \leq t} (M(t) - M(s) + R(s)) \]
2. MORE EXAMPLES
A System with Loss [Chuang and Cheng 2000]

- node with service curve $\beta(t)$ and buffer of size $X$
- when buffer is full incoming data is discarded
- modelled by a virtual controller (not buffered)
- fluid model or fixed sized packets
- Pb: find loss ratio
A System with Loss

Assume $R$ is $\alpha$ – smooth; if $X \geq \nu(\alpha, \beta)$ then no loss

If $X < h(\alpha, \beta)$, what can we say?
Thm [Chuang and Cheng 2000] Let $r$ be the largest such that $X = v(r\alpha, \beta)$ i.e. $r = 1 \wedge \inf_{t>0} \left( \frac{\beta(t)+X}{\alpha(t)} \right)$

Then $\frac{L(t)}{R(t)} \leq 1 - r^*$; it is the best possible bound.
Analysis of System with Loss

1. $R'(t) - R'(s) \leq R(t) - R(s) \forall s \leq t$ (splitter)
2. $R'(t) - \Pi R'(t) \leq X$ (buffer does not overflow)

where $\Pi$ is the transformation $R' \rightarrow R$, assumed isotone and usc ("physical assumptions")

There is a maximum solution and $R'$ is the maximum solution
Analysis of System with Loss

1. $R'(t) - R'(s) \leq R(t) - R(s) \forall s \leq t$ (splitter)
2. $R'(t) - \Pi R'(t) \leq X$ (buffer does not overflow)

- Let $x(t) = rR(t)$ with $r$ given by thm.
- Eqn 1 is satisfied
- $x$ is $r\alpha$ — smooth, thus required buffer $\leq X$ and Eqn 2 is satisfied
- Thus $R'(t) \geq x(t)$ and
  
  \[
  \frac{L(t)}{R(t)} = 1 - \frac{R'(t)}{R(t)} \leq 1 - \frac{x(t)}{R(t)} = 1 - r
  \]
Optimal Smoothing [L., Verscheure 2000]

- Network + end-client offer a service curve $\beta$ to flow $R'(t)$
- Smoother delivers a flow $R'(t)$ conforming to an arrival curve $\sigma$.
- Video stream is stored in the client buffer, read after a playback delay $D$.
- Pb: which smoothing strategy minimizes $D$?
Optimal Smoothing, System Equations

- (1) $R'$ is $\sigma$-smooth
- (2) $(R' \otimes \beta)(t) \geq R(t-D)$

Use deconvolution $(a \ominus b)(t) = \sup_{s \geq 0} (a(t + s) - b(s))$

$x \leq y \otimes \beta \iff x \ominus \beta \leq y$

- system becomes
  - (1) $R' \geq R' \ominus \sigma$
  - (2) $R' \geq (R \ominus \beta)(t-D)$
Optimal Smoothing, System Equations

This is a max-plus linear problem, it has a minimum solution $R'$ given by the iterations:

1. $R' \geq R' \ominus \sigma$
2. $R' \geq (R \ominus \beta)(t-D)$

Thus $R'(t) = (R \ominus (\sigma \otimes \beta))(t-D)$
Example

A possible $R'(t)$

$R \odot (\sigma \otimes \beta)(t-D)$

$\sigma \otimes \beta(t)$

$R(t-D)$
Minimum Playback Delay

D must satisfy:
\[ R \ominus (\beta \otimes \sigma) (-D) \geq 0 \]

this is equivalent to
\[ D \geq h(R, \beta \otimes \sigma) \]
\( R(t) \)

\[ (\sigma \otimes \beta)(t) \]

\( D = 435 \text{ ms} \)

\[ (\sigma \otimes \beta)(t) \]

\( D = 102 \text{ ms} \)
The Perfect Battery

- Battery may be charged \((u(t) > \ell(t))\) or discharged \((u(t) < \ell(t))\)
- Load \(\ell(t)\) is given
- Problem is to determine a power schedule \(u(t)\), subject to \(0 \leq u(t) \leq g(t)\) and within battery constraints
System Equations for the Perfect Battery

1. $L(t) \leq B_0 + U(t)$ no underflow
2. $U(t) - L(t) + B_0 \leq B$ no overflow
3. $U(t) - U(s) \leq G(t) - G(s), \forall s \leq t$ power constraint

where $U(t), L(t), G(t)$ are cumulative functions such as $U(t) = \int_0^t u(s) ds$
System Equations

1. \( L(t) \leq B_0 + U(t) \) no underflow
2. \( U(t) - L(t) + B_0 \leq B \) no overflow
3. \( U(t) - U(s) \leq G(t) - G(s), \forall s \leq t \)

- Relax (eq 1):
  \[
  U(t) \leq (B - B_0 + L(t))1_{t>0} \\
  U(t) \leq h_G(U)(t)
  \]

There is a maximum solution,
\[
U^*(t) = G(t) \land \inf_{s \leq t}(G(t) - G(s) + L(s) + B - B_0)
\]

\( U^* \) is causal

The problem is feasible iff \( U^* \) satisfies (eq 1), i.e.
\[
\begin{align*}
  B_0 &\geq \sup_t (L(t) - G(t)) \\
  B &\geq \sup_{0 \leq s \leq t} (L(t) - L(s) - G(t) + G(s))
\end{align*}
\]
System Equations

1. $L(t) \leq B_0 + U(t)$ no underflow
2. $U(t) - L(t) + B_0 \leq B$ no overflow
3. $U(t) - U(s) \leq G(t) - G(s), \forall s \leq t$

Relax (eq 2):

\[
U(t) \geq \max(0, -B_0 + L(t)) \\
U(s) \geq \sup_{\tau \geq s} (G(s) - G(\tau) + U(\tau))
\]

There is a minimum solution,

\[
U_*(t) = 0 \lor \sup_{\tau \geq t} (G(t) - G(\tau) + L(\tau) - B_0)
\]

$U_*$ is non-causal

The problem is feasible iff $U_*$ satisfies (eq 2)

This gives the same conditions
3. TIME VERSUS SPACE
The Residuation Theorem is a Space Method

- The maximum solution $x^*$ to the problem
  \[ x \leq b \\
  x \leq \Pi x \]
  is given by iterates over the entire trajectory
  \[
  x^{(0)} = b \\
  x^{(1)} = \Pi x^{(0)} \\
  x^{(2)} = \Pi x^{(1)} \\
  etc
  \]

- When time is discrete there may be another way to compute $x^*$ by time recursion
The Shaper, Time Method

- Time is discrete \( t = 0, 1, 2, \ldots \)
- Define \( R' \) by:
  \[
  R'(0) = R(0) = 0 \\
  R'(t) = R(t) \land \inf_{0 \leq u \leq t-1} (\sigma(t-u) + R'(u))
  \]
- \( R' \) is solution
- For any other solution \( x, x(t) \leq R'(t) \) [induction]
- \( R' \) is the maximal solution, i.e. \( R' = R^* \).
- Note the difference in representation:
  \[
  R^*(t) = R(t) \land \inf_{0 \leq u \leq t-1} (\sigma(t-u) + R(u))
  \]

\[
(1) \ x \leq x \otimes \sigma \\
(2) \ x \leq R
\]
The Time Method for *Linear* Problems

- [L., Thiran 2001] Thm 4.4.1: the problem in discrete time
  \[ x(t) \leq b(t) \]
  \[ x(t) \leq \inf_{s} (H(t, s) + x(s)) \]
  where \( H : N \times N \rightarrow R^+ \), ↑ in \( t \) and ↓ in \( s \)

has a maximal solution \( x^* \) given by
  \[ x^*(0) = b(0) \]
  \[ x^*(t) = x(t) \wedge \inf_{0 \leq u \leq t-1} (H(t, u) + x^*(u)) \]

- This is a second, alternative representation for \( x^* \)
There is a maximum solution,

\[ U^*(t) = G(t) \land \inf_{s \leq t} (G(t) - G(s) + L(s) + B - B_0) \]

It can be computed by the time method:

\[ u^*(t) = \min (g(t), B - B(t) + \ell(t)) \]

The minimum schedule is anti-causal and can be computed with time reversal
Conclusion

- Min-plus and max-plus system theory contains a central result: residuation theorem (= fixed point theorem)
  Establishes existence of maximum (resp. minimum) solutions
  and provides a representation

- Space and Time methods give different representations
Thank You...