ODE and Discrete Simulation
or
Mean Field Methods for Computer and Communication Systems

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MEAN FIELD INTERACTION MODEL
Mean Field

A *model* introduced in Physics

- interaction between *particles* is via distribution of states of all particles

An *approximation* method for a large collection of particles

- assumes *independence* in the master equation

Why do we care in I&C?

- Model interaction of many objects:
  - Distributed systems, communication protocols, game theory, self-organized systems
Mean Field Interaction Model

- Time is discrete
- \( N \) objects, \( N \) large
- Object \( n \) has state \( X_n(t) \)
- \( (X^N_1(t), \ldots, X^N_N(t)) \) is Markov
- Objects are observable only through their state

“Occupancy measure”
\( M^N(t) = \text{distribution of object states at time } t \)
Mean Field Interaction Model

- Time is discrete
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- Objects are observable only through their state

- "Occupancy measure" $M^N(t)$ = distribution of object states at time $t$
- **Theorem** [Gast (2011)] $M^N(t)$ is Markov
- Called "Mean Field Interaction Models" in the Performance Evaluation community [McDonald (2007), Benaïm and Le Boudec (2008)]
A Few Examples Where Applied


Example: 2-Step Malware

- Mobile nodes are either
  - `S` Susceptible
  - `D` Dormant
  - `A` Active
- Time is discrete
- Nodes meet pairwise (bluetooth)
- One interaction per time slot, $I(N) = 1/N$; mean field limit is an ODE
- State space is finite
  $= \{`S',`A',`D'\}$
- Occupancy measure is
  $M(t) = (S(t), D(t), A(t))$ with
  $S(t) + D(t) + A(t) = 1$
  $S(t) = \text{proportion of nodes in state `S'}$
  [Benaïm and Le Boudec(2008)]

Possible interactions:

1. Recovery
   - $D \rightarrow S$
2. Mutual upgrade
   - $D + D \rightarrow A + A$
3. Infection by active
   - $D + A \rightarrow A + A$
4. Recovery
   - $A \rightarrow S$
5. Recruitment by Dormant
   - $S + D \rightarrow D + D$
   Direct infection
   - $S \rightarrow D$
6. Direct infection
   - $S \rightarrow A$
Simulation Runs, N=1000 nodes

\[ A(t) \] Proportion of nodes in state \( i=2 \)

\[ D(t) \] Proportion of nodes in state \( i=1 \)

\[ \beta = 0.01, \delta_A = 0.005, \delta_D = 0.0001, \alpha_0 = \alpha = 0.0001, h = 0.3, r = 0.1, \lambda = 0.0001 \]
Sample Runs with $N = 1000$

\[\beta = 0.01, \delta_A = 0.005, \delta_D = 0.0001, \alpha_0 = \alpha = 0.0001, h = 0.3, r = 0.1, \lambda = 0.0001\]
Example: TCP and ECN

- [Tinnakornsrisuphap and Makowski(2003)]

- Time is discrete, mean field limit is also in discrete time (iterated map)

- Similar examples:
  - HTTP Metastability
    [Baccelli et al.(2004)Baccelli, Lelarge, and McDonald]
  - Reputation System [Le Boudec et al.(2007)Le Boudec, McDonald, and Mundinger]

At, every time step, all connections update their state: \( I(N) = 1 \)
The Importance of Being Spatial

- Mobile node state = (c, t)
  c = 1 ... 16 (position)
  t ∈ R⁺ (age of gossip)

- Time is continuous, I(N) = 1

- Occupancy measure is
  \( F_c(z,t) = \text{proportion of nodes that at location } c \text{ and have age } \leq z \)

[Age of Gossip, Chaintreau et al. (2009)]
What can we do with a Mean Field Interaction Model?

- **Large $N$ asymptotics, Finite Horizon**
  - fluid limit of occupancy measure (ODE)
  - decoupling assumption (fast simulation)

- **Issues**
  - When valid
  - How to formulate the fluid limit

- **Issues**
  - When valid

- **Large $t$ asymptotic**
  - Stationary approximation of occupancy measure
  - Decoupling assumption
2.

CONVERGENCE TO ODE
Intensity $I(N)$

- $I(N) = \text{expected number of transitions per object per time unit}$

- A mean field limit occurs when we re-scale time by $I(N)$
  i.e. we consider $X^N(t/I(N))$

- $I(N) = O(1)$: mean field limit is in discrete time
  [Le Boudec et al (2007)]

- $I(N) = O(1/N)$: mean field limit is in continuous time
  [Benaïm and Le Boudec (2008)]
The Mean Field Limit

Under very general conditions (given later) the occupancy measure converges, in law, to a deterministic process, $m(t)$, called the mean field limit

$$M^N \left( \frac{t}{I(N)} \right) \rightarrow m(t)$$

Finite State Space => ODE
Mean Field Limit
$N = +\infty$

Stochastic system
$N = 1000$
Sufficient Conditions for Convergence

- [Kurtz 1970], see also [Bordenav et al 2008], [Graham 2000]

Sufficient condition verifiable by inspection:

[Benaïm and Le Boudec(2008), Ioannidis and Marbach(2009)]

Let $W^N(k)$ be the number of objects that do a transition in time slot $k$. Note that $\mathbb{E}(W^N(k)) = NI(N)$, where $I(N) \overset{def}{=} \text{intensity}$. Assume

$$\mathbb{E}\left(W^N(k)^2\right) \leq \beta(N) \quad \text{with} \quad \lim_{N \to \infty} I(N)\beta(N) = 0$$

Example: $I(N) = 1/N$

Second moment of number of objects affected in one timeslot = $o(N)$

Similar result when mean field limit is in discrete time [Le Boudec et al 2007]
Rescale time such that one time step $= 1/N$

Number of transitions per time step is bounded by 2, therefore there is convergence to mean field

\[
\frac{\partial D}{\partial t} = -\delta_D D - 2\lambda D^2 - \beta A \frac{D}{h + D} + (\alpha_0 + rD)S
\]

\[
\frac{\partial A}{\partial t} = 2\lambda D^2 + \beta A \frac{D}{h + D} - \delta_A A + \alpha S
\]

\[
\frac{\partial S}{\partial t} = \delta_D D + \delta_A A - (\alpha_0 + rD)S - \alpha S
\]
Formulating the Mean Field Limit

- **Drift** = sum over all transitions of proba of transition $x$ Delta to system state $M^N(t)$

- Re-scale drift by intensity

- Equation for mean field limit is

  $\frac{dm}{dt} = \text{limit of rescaled drift}$

- Can be automated

  http://icawww1.epfl.ch/IS/tsed

<table>
<thead>
<tr>
<th>case</th>
<th>prob</th>
<th>effect on $(D, A, S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$D\delta_D$</td>
<td>$\frac{1}{N}(-1, 0, 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$D\lambda\frac{N-1}{N-1}$</td>
<td>$\frac{1}{N}(-2, +2, 0)$</td>
</tr>
<tr>
<td>3</td>
<td>$A\beta\frac{D}{h+D}$</td>
<td>$\frac{1}{N}(-1, +1, 0)$</td>
</tr>
<tr>
<td>4</td>
<td>$A\delta_A$</td>
<td>$\frac{1}{N}(0, -1, +1)$</td>
</tr>
<tr>
<td>5</td>
<td>$S(\alpha_0 + rD)$</td>
<td>$\frac{1}{N}(+1, 0, -1)$</td>
</tr>
<tr>
<td>6</td>
<td>$S\alpha$</td>
<td>$\frac{1}{N}(0, +1, -1)$</td>
</tr>
</tbody>
</table>

$$drift = \frac{1}{N} \left( -D\delta_D - 2D\lambda\frac{N-1}{N-1} - A\beta\frac{D}{h+D} + S(\alpha_0 + rD) \right)$$

$$\frac{\partial D}{\partial t} = -\delta_D D - 2\lambda D^2 - \beta A\frac{D}{h+D} + (\alpha_0 + rD)S$$

$$\frac{\partial A}{\partial t} = 2\lambda D^2 + \beta A\frac{D}{h+D} - \delta_A A + \alpha S$$

$$\frac{\partial S}{\partial t} = \delta_D D + \delta_A A - (\alpha_0 + rD)S - \alpha S$$
Convergence to Mean Field

For the finite state space case, there are many simple results, often verifiable by inspection.

For example [Kurtz 1970] or [Benaim, Le Boudec 2008]

For the general state space, things may be more complex (fluid limit is not an ODE, e.g. [Chaintreau et al, 2009])
3.

FINITE HORIZON: FAST SIMULATION AND DECOUPLING ASSUMPTION
Convergence to Mean Field Limit is Equivalent to Propagation of Chaos

**Definition 1.1** Let $X^N = (X_1^N, ..., X_N^N)$ be an exchangeable sequence of processes in $\mathcal{P}(S)$ and $m \in \mathcal{P}(S)$ where $S$ is metric complete separable. $(X^N)_N$ is $m$-chaotic iff for every $k$: $\mathcal{L}(X_1^N, ..., X_k^N) \to m \otimes ... \otimes m$ as $N \to \infty$.

**Theorem 1.1 ([Sznitman(1991)])** $(X^N)_N$ is $m$-chaotic then the occupancy measure $M^N \overset{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n^N}$ converges in probability (and in law) to $m$.

If the occupancy measure converges in law to $m$ then $(X^N)_N$ is $m$-chaotic.
Propagation of Chaos = Decoupling Assumption

(Propagation of Chaos)

$k$ objects are asymptotically independent with common law equal to the mean field limit, for any fixed $k$

$$\mathcal{L} \left( X_1 \left( \frac{t}{I(N)} \right), ..., X_k \left( \frac{t}{I(N)} \right) \right) \rightarrow m(t) \otimes ... \otimes m(t)$$

(Decoupling Assumption)

(also called Mean Field Approximation, or Fast Simulation)

The law of one object is asymptotically as if all other objects were drawn randomly with replacement from $m(t)$
The Two Interpretations of the Mean Field Limit

- At any time $t$

\[
P(X_n(t) = A') \approx A\left(\frac{t}{N}\right)
\]
\[
P(X_m(t) = D', X_n(t) = A') \approx D\left(\frac{t}{N}\right) A\left(\frac{t}{N}\right)
\]
where $(D, A, S)$ is solution of ODE

- Thus for large $t$:
  - $\text{Prob (node } n \text{ is dormant)} \approx 0.3$
  - $\text{Prob (node } n \text{ is active)} \approx 0.6$
  - $\text{Prob (node } n \text{ is susceptible)} \approx 0.1$

- $m(t)$ approximates both
  1. the occupancy measure $M^N(t)$
  2. the state probability for one object at time $t$, drawn at random among $N$
\( p_j^N(t|i) \) is the probability that a node that starts in state \( i \) is in state \( j \) at time \( t \):

\[
p_j^N(t|i) = \mathbb{P}(X_n^N(t) = j | X_n^N(0) = i)
\]

Then \( p_j^N(t/N|i) \approx p_j(t|i) \) where \( p(t|i) \) is a continuous time, non homogeneous process

\[
\frac{d}{dt} \bar{p}(t|i) = \bar{p}(t|i)^T A(\bar{m}(t)))
\]

\[
\frac{d}{dt} \bar{m}(t) = \bar{m}(t)^T A(\bar{m}(t))) = F(\bar{m}(t))
\]

Same ODE as mean field limit, but with different initial condition
The Decoupling Assumption

- The evolution for one object as if the other objects had a state drawn randomly and independently from the distribution $m(t)$.

- Is valid over finite horizon whenever mean field convergence occurs.

- Can be used to analyze or simulate evolution of $k$ objects.
4. INFINITE HORIZON: FIXED POINT METHOD AND DECOUPLING ASSUMPTION
The Fixed Point Method

Decoupling assumption says distribution of prob for state of one object is approx. \( m(t) \) with \( \frac{d\vec{m}}{dt} = F(\vec{m}) \)

We are interested in stationary regime, i.e. we do \( F(\vec{m}) = \vec{0} \)

This is the « Fixed Point Method »

Example: in stationary regime:
- Prob (node \( n \) is dormant) \( \approx 0.3 \)
- Prob (node \( n \) is active) \( \approx 0.6 \)
- Prob (node \( n \) is susceptible) \( \approx 0.1 \)
- Nodes \( m \) and \( n \) are independent
Example: 802.11 Analysis, Bianchi’s Formula

802.11 single cell
\[ m_i = \text{proba one node is in backoff stage I} \]
\[ \beta = \text{attempt rate} \]
\[ \gamma = \text{collision proba} \]

See [Benaim and Le Boudec, 2008] for this analysis

ODE for mean field limit
\[
\frac{dm_0}{d\tau} = -m_0q_0 + \beta(\bar{m})(1 - \gamma(\bar{m})) + q_K m_K \gamma(\bar{m})
\]
\[
\frac{dm_i}{d\tau} = -m_iq_i + m_{i-1}q_{i-1} \gamma(\bar{m}) \quad i = 1, \ldots, K
\]

\[ \beta(\bar{m}) = \sum_{i=0}^{K} q_i m_i \]
\[ \gamma(\bar{m}) = 1 - e^{-\beta(\bar{m})} . \]

Solve for Fixed Point:

\[
m_i = \frac{\gamma^i}{q_i \sum_{k=0}^{K} \frac{\gamma^k}{q_k}} \]

Bianchi’s Fixed Point Equation [Bianchi 1998]

\[
\gamma = 1 - e^{-\beta}
\]
\[
\beta = \frac{\sum_{k=0}^{K} \gamma^k}{\sum_{k=0}^{K} \frac{\gamma^k}{q_k}}
\]
2-Step Malware, Again

- Same as before except for one parameter value: $h = 0.1$ instead of 0.3

- The ODE does not converge to a unique attractor (limit cycle)

- The equation $F(m) = 0$ has a unique solution (red cross) – but it is not the stationary regime!
Example Where Fixed Point Method Fails

- In stationary regime, \( m(t) = (D(t), A(t), S(t)) \) follows the limit cycle
- Assume you are in stationary regime (simulation has run for a long time) and you observe that one node, say \( n=1 \), is in state ‘A’
- It is more likely that \( m(t) \) is in region \( R \)
- Therefore, it is more likely that some other node, say \( n=2 \), is also in state ‘A’

- This is synchronization
Joint PDFs of Two Nodes in Stationary Regime

Mean of Limit of $\mathbf{\pi}^N = \text{pdf of one node in stationary regime}$

Stationary point of ODE

pdf of node 2 in stationary regime, given node 1 is A

pdf of node 2 in stationary regime, given node 1 is S

pdf of node 2 in stationary regime, given node 1 is D
Where is the Catch?

- Decoupling assumption says that nodes $m$ and $n$ are asymptotically independent.

- There is mean field convergence for this example.

- But we saw that nodes may not be asymptotically independent.

...is there a contradiction?
The **decoupling assumption may not hold in stationary regime**, even for perfectly regular models.
Result 1: Fixed Point Method Holds under (H)

Assume that

(H) ODE has a unique global stable point to which all trajectories converge

Theorem [e.g. Benaim et al 2008]: The limit of stationary distribution of $M^N$ is concentrated on this fixed point

The decoupling assumption holds in stationary regime
Here:
Birkhoff center = limit cycle \(\cup\) fixed point

Theorem in [Benaim] says that the stochastic system for large \(N\) is close to the Birkhoff center,

i.e. the stationary regime of ODE is a good approximation of the stationary regime of stochastic system
Stationary Behaviour of Mean Field Limit is not predicted by Structure of Markov Chain

- $M^N(t)$ is a Markov chain on $S^N = \{(a, b, c) \geq 0, a + b + c = 1, \ a, b, c$ multiples of $1/N\}$
- $M^N(t)$ is ergodic and aperiodic

Depending on parameter, there is or is not a limit cycle for $m(t)$

$h = 0.3$

$h = 0.1$
Example: 802.11 with Heterogeneous Nodes

[Cho et al, 2010]

Two classes of nodes with heterogeneous parameters (retransmission probability)

Fixed point equation has a unique solution, but this is not the stationary proba

There is a limit cycle
Result 3: In the Reversible Case, the Fixed Point Method Always Works

**Definition** Markov Process $X(t)$ on enumerable state $E$ space, with transition rates $q(i,j)$ is reversible iff

1. it is ergodic  
2. $p(i) q(i,j) = p(j) q(j,i)$ for some $p$

**Theorem 1.2 ([Le Boudec(2010)])** Assume some process $Y^N(t)$ converges at any fixed $t$ to some deterministic system $y(t)$ at any finite time. Assume the processes $Y^N$ are reversible under some probabilities $\Pi^N$. Let $\Pi \in \mathcal{P}(E)$ be a limit point of the sequence $\Pi^N$. $\Pi$ is concentrated on the set of stationary points $S$ of the fluid limit $y(t)$

- Stationary points = fixed points
- If process with finite $N$ is reversible, the stationary behaviour is determined only by fixed points.
A Correct Method

1. Write dynamical system equations \textit{in transient regime}

2. Study the \textit{stationary regime of} dynamical system
   - \textbf{if} converges to unique stationary point \( m^* \)
   - \textbf{then} make fixed point assumption
   - \textbf{else} objects are coupled in stationary regime by mean field limit \( m(t) \)

Hard to predict outcome of 2 (except for reversible case)
Conclusion

- Mean field models are frequent in large scale systems

- Validity of approach is often simple by inspection

- Mean field is both
  - ODE for fluid limit
  - Fast simulation using decoupling assumption

- Decoupling assumption holds at finite horizon; may not hold in stationary regime.

- Stationary regime is more than stationary points, in general (except for reversible case)
Thank You ...


